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LETTER TO THE EDITOR

Dirac quantisation of a massive spin-$\frac{3}{2}$ particle coupled with a magnetic field

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Abstract. The canonical quantisation of the massive spin-$\frac{3}{2}$ field coupled with the external electromagnetic field is carried out by the use of Dirac's Hamiltonian method. It is shown that the quantisation can be achieved even when the magnetic field has such a strength that the secondary constraint equation cannot be employed freely. In this case, the equal-time anticommutation relation between the field operator and its Hermite conjugate contains the fourth-order derivatives of the three-dimensional delta function.

Recently, Dirac quantisation (Dirac 1964) of free massive spin-$\frac{3}{2}$ and spin-2 fields has been examined by Baaklini and Tuite (1978, 1979). On the other hand, Johnson and Sudarshan (1961) pointed out that there exists an inconsistency when one quantises the Rarita–Schwinger field interacting with the electromagnetic field within the framework of a positive definite metric. We think that the difficulty comes from the fact that the secondary constraint equation cannot be used freely when the magnitude of the magnetic field $\boldsymbol{H}$ has a special value. It is therefore interesting to investigate Dirac quantisation of the spin-$\frac{3}{2}$ field coupled with the magnetic field of this special strength. This is the aim of the present letter.

In order to simplify the discussion we confine ourselves to the case where the electromagnetic field is treated as an external one, and start with the following Lagrangian density

$$L=\tfrac{1}{2}\epsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma_5\gamma_\mu\partial_\nu\psi_\rho-\tfrac{1}{2}\epsilon^{\lambda\rho\mu\nu}(\partial_\nu\bar{\psi}_\lambda)\gamma_5\gamma_\mu\psi_\rho$$
$$+\mathrm{i}m\bar{\psi}_\lambda\sigma^{\lambda\rho}\psi_\rho-\mathrm{i}e\epsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma_5\gamma_\mu\psi_\rho A_\nu. \tag{1}$$

Our notations are $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, $\gamma_5=\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$, $\sigma^{\mu\nu}=\frac{1}{2}\mathrm{i}[\gamma^\mu,\gamma^\nu]$, $\gamma^{\mu\dagger}=\gamma_0\gamma^\mu\gamma_0$ and $\epsilon^{0123}=1$.

The momenta conjugate to $(\psi_\lambda)_\alpha$ and $(\psi_\lambda^\dagger)_\alpha$ are

$$\begin{aligned}\pi_\alpha^0&=0, &\quad \pi_\alpha^k&=\tfrac{1}{2}(\epsilon^{kij}\psi_i^\dagger\gamma_5\gamma_j\gamma_0)_\alpha\\ \pi_\alpha^{0\dagger}&=0, &\quad \pi_\alpha^{k\dagger}&=\tfrac{1}{2}(\epsilon^{kij}\gamma_0\gamma_5\gamma_i\psi_j)_\alpha.\end{aligned} \tag{2}$$

The fundamental Poisson brackets (PB's) are

$$(\psi_{\mu\alpha}(x),\pi_\beta^\nu(y))=(\psi_{\mu\alpha}^\dagger(x),\pi_\beta^{\nu\dagger}(y))=\delta_\mu^\nu\delta_{\alpha\beta}\delta^3(x-y) \tag{3}$$

and the others are zero. The general Hamiltonian is defined by

$$H^*=H+u_{\mu\alpha}\phi_\alpha^\mu+u_{\mu\alpha}^\dagger\phi_\alpha^{\mu\dagger} \tag{4}$$

where

$$H = -\tfrac{1}{2}\epsilon^{\lambda\rho\mu k}\bar{\psi}_\lambda\gamma_5\gamma_\mu\partial_k\psi_\rho + \tfrac{1}{2}\epsilon^{\lambda\rho\mu k}(\partial_k\bar{\psi}_\lambda)\gamma_5\gamma_\mu\psi_\rho - im\bar{\psi}_\lambda\sigma^{\lambda\rho}\psi_\rho + ie\epsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma_5\gamma_\mu\psi_\rho A_\nu \tag{5}$$

and ϕ_α^μ and $\phi_\alpha^{\mu\dagger}$ are the primary constraints

$$\begin{aligned} \phi_\alpha^0 &\equiv \pi_\alpha^0 = 0, & \phi_\alpha^k &\equiv \pi_\alpha^k - \tfrac{1}{2}(\epsilon^{kij}\psi_i^\dagger\gamma_5\gamma_j\gamma_0)_\alpha = 0 \\ \phi_\alpha^{0\dagger} &\equiv \pi_\alpha^{0\dagger} = 0, & \phi_\alpha^{k\dagger} &\equiv \pi_\alpha^{k\dagger} - \tfrac{1}{2}(\epsilon^{kij}\gamma_0\gamma_5\gamma_i\psi_j)_\alpha = 0. \end{aligned} \tag{6}$$

The conditions that $\phi_\alpha^k = 0$ persist in time determine $u_{k\alpha}$, while $\phi_\alpha^0 = 0$ yields the secondary constraints

$$\chi \equiv \sigma^{ij}D_i\psi_j + m\gamma^i\psi_j = 0, \qquad \bar{\chi} \equiv D_i^*\bar{\psi}_j\sigma^{ij} + m\bar{\psi}_j\gamma^j = 0 \tag{7}$$

with $D_i = \partial_i - ieA_i$ and $D_i^* = \partial_i + ieA_i$. From the consistency condition $d\chi/dx^0 = (\chi, H^*) = 0$ and $d\bar{\chi}/dx^0 = 0$, we have the secondary constraints

$$\theta \equiv 2R\gamma_0\psi_0 + \Gamma^k\psi_k = 0, \qquad \bar{\theta} \equiv 2\bar{\psi}_0\gamma^0 R + \bar{\psi}_k\Gamma^k = 0 \tag{8}$$

where

$$R \equiv \tfrac{1}{2}[1 - (e/3m^2)\sigma^{ij}F_{ij}], \qquad \Gamma^k \equiv \gamma^k + (ie/3m^2)\gamma^\lambda\gamma^k\gamma^\rho F_{\lambda\rho}. \tag{9}$$

If

$$\det R = (\tfrac{1}{2})^4[1 - (2e/3m^2)^2\boldsymbol{H}^2]^2 \tag{10}$$

does not vanish, the consistency conditions for $\theta = \bar{\theta} = 0$ determine u_0 and no more constraints are generated. All the constraints are of the second class. On the contrary, it becomes complicated when $\det R = 0$. The latter case is dealt with later. The constraints $\phi^0 = \phi^{0\dagger} = 0$ and $\theta = \bar{\theta} = 0$ are regarded as defining equations for ψ_0 in terms of ψ_i.

The Dirac PB's are defined by the following two steps. In the first step, by using the second-class constraints $\phi_\alpha^i = 0$ and $\phi_\beta^{j\dagger} = 0$, Dirac PB's among $\psi_{\mu\alpha}(x)$ and $\psi_{\nu\beta}^\dagger(y)$ are defined in a way similar to the case of free field (Baaklini and Tuite 1978):

$$(\psi_{i\alpha}(x), \psi_{j\beta}^\dagger(y))^* = \tfrac{1}{2}i(\gamma_j\gamma_i)_{\alpha\beta}\delta^3(x-y), \tag{11}$$

the others being zero. Using the brackets

$$\begin{aligned} &(\chi_\alpha(x), \chi_\beta(y))^* = (\bar{\chi}_\alpha(x), \bar{\chi}_\beta(y))^* = 0 \\ &(\chi_\alpha(x), \bar{\chi}_\beta(y))^* = 3im^2(\gamma_0 R)_{\alpha\beta}\delta^3(x-y) \end{aligned} \tag{12}$$

which are defined by means of (11), we define the second step Dirac PB for any dynamical variables $f(x)$ and $g(y)$ by

$$(f(x), g(y))^{**} \equiv (f(x), g(y))^* - \iint d^3z_1\, d^3z_2 (f(x), \chi_m(z_1))^* D_{mn}^{-1}(z_1, z_2)(\chi_n(z_2), g(y))^* \tag{13}$$

in which χ_m denote χ_α and $\bar{\chi}_\beta$ and $D_{mn} = (\chi_m, \chi_n)^*$. As shown by two of us (Endo and Kimura 1979), the bracket with the double asterisk is the same as the Dirac PB calculated in a single step. Thus, the final Dirac PB's among $\psi_{\mu\alpha}(x)$ and $\psi^{\dagger}_{\nu\beta}(y)$ are given by

$$(\psi_{i\alpha}(x), \psi^{\dagger}_{j\beta}(y))_D$$
$$= [(\mathrm{i}/2)\gamma_j\gamma_i + (\mathrm{i}/12)\gamma_i R^{-1}\gamma_j - (1/6m)\gamma_i R^{-1} D_j + (1/6m) R^{-1}\gamma_j D_i + (\mathrm{i}/3m^2) R^{-1} D_i D_j]_{\alpha\beta}\delta^3(x-y) \tag{14}$$
$$(\psi_{0\alpha}(x), \psi^{\dagger}_{j\beta}(y))_D = -\tfrac{1}{2}(R^{-1}\gamma_0\Gamma^i)_{\alpha\gamma}(\psi_{i\gamma}(x), \psi^{\dagger}_{j\beta}(y))_D$$
$$(\psi_{0\alpha}(x), \psi^{\dagger}_{0\beta}(y))_D = \tfrac{1}{4}(R^{-1}\gamma_0\Gamma^i)_{\alpha\gamma}(\psi_{i\gamma}(x), \psi^{\dagger}_{j\delta}(y))_D(\gamma_0\Gamma^j R^{-1})_{\delta\beta}.$$

The quantisation is performed by equating the equal-time anti-commutation relation $\{\psi_{\mu\alpha}(x), \psi^{\dagger}_{\nu\beta}(y)\}$ to $\mathrm{i}(\psi_{\mu\alpha}(x), \psi^{\dagger}_{\nu\beta}(y))_D$. It is shown that the Heisenberg equations of motion

$$\mathrm{i}\dot{\psi}_{i\alpha}(x) = [\psi_{i\alpha}(x), H] \tag{15}$$

coincide with the Lagrangian equations of motion. The anticommutation relation $\{\gamma^i\psi_i, \psi^{\dagger}_j\gamma^j\}$ is the same as that of Johnson and Sudarshan.

We shall now enter into the main problem in which the magnetic field $\boldsymbol{H}$ satisfies $\det R = 0$. Here, we take $F_{0i} = 0$ and $F_{ij} =$ constant in order to simplify the calculation. We introduce $\tilde{R}$ by

$$\tilde{R} \equiv 1 - R = \tfrac{1}{2}\{1 + (e/3m^2)\sigma^{ij}F_{ij}\}. \tag{16}$$

The R and $\tilde{R}$ satisfy the relationship

$$R\tilde{R} = 0 \tag{17}$$

and their rank is two. Making projections of θ and $\bar{\theta}$ by R and $\tilde{R}$, we have

$$R\theta \equiv 2R\gamma_0\psi_0 + R\Gamma^k\psi_k = 0, \qquad \bar{\theta}R \equiv 2\bar{\psi}_0\gamma_0 R + \bar{\psi}_k\Gamma^k R = 0 \tag{18}$$
$$\tilde{R}\theta \equiv \tilde{\theta} \equiv \tilde{R}\Gamma^k\psi_k = 0, \qquad \bar{\theta}\tilde{R} \equiv \tilde{\bar{\theta}} \equiv \bar{\psi}_k\Gamma^k\tilde{R} = 0. \tag{19}$$

The consistency conditions $R\theta = \bar{\theta}R = 0$ determine Ru_0 and $u_0^{\dagger}R$, while $\mathrm{d}\tilde{\theta}/\mathrm{d}x^0 = \mathrm{d}\tilde{\bar{\theta}}/\mathrm{d}x^0 = 0$ give rise to

$$\xi \equiv \tilde{R}\gamma_0\psi_0 + \tilde{R}\Lambda^k\psi_k = 0, \qquad \xi^{\dagger} \equiv \psi_0^{\dagger}\gamma_0\tilde{R} + \psi_k^{\dagger}\Lambda^{k\dagger}\tilde{R} \tag{20}$$

where

$$\Lambda^k \equiv \gamma^k + (\mathrm{i}/2m)(2e/3m^2)^2 F_{lm}D^l\gamma^m F_{ij}\gamma^i g^{jk} + (2/m)(2e/3m^2)\times F_{ij}D^i g^{jk} \tag{21}$$

and $D^{l\dagger} = \partial^l + \mathrm{i}eA^l$. The consistency conditions $\mathrm{d}\xi/\mathrm{d}x^0 = \mathrm{d}\xi^{\dagger}/\mathrm{d}x^0 = 0$ determine $\tilde{R}u_0$ and $u_0^{+}\tilde{R}$.

All the constraints become the second class. The constraints $\phi^0 \equiv \pi^0 = 0$ and $\frac{1}{2}R\theta + \xi = 0$ and their Hermite conjugates determine ψ_0 and $\psi_0^{\dagger}$ in terms of ψ_k and $\psi_k^{\dagger}$. The procedure of the first step to define Dirac PB is the same as in the case of $\det R \neq 0$. On the contrary, the procedure of the second step becomes cumbersome in the present case. We shall define brackets with double asterisks by taking χ_α, $\bar{\chi}_\beta$, $\tilde{\theta}$ and $\tilde{\bar{\theta}}$ as the second-class constraints $\chi_m (m = 1, 2, \ldots, 12)$. To calculate the inverse of $D_{mn}(z_1, z_2) = (\chi_m(z_1), \chi_n(z_2))^*$ it is convenient to adopt a special coordinate system in which the magnetic field $\boldsymbol{H}$ is $\boldsymbol{H}=(0,0,3m^2/2e)$. In the Dirac representation of γ^μ, the result is

$$D^{-1}_{mn}(z_1,z_2)=\begin{matrix} \\ \chi \\ \tilde{\theta} \\ \bar{\chi} \\ \tilde{\bar{\theta}} \end{matrix}\overset{\begin{matrix}\chi & \tilde{\theta} & \bar{\chi} & \tilde{\bar{\theta}}\end{matrix}}{\begin{pmatrix} 0 & V^T \\ V & 0 \end{pmatrix}}\delta^3(z_1-z_2) \tag{22}$$

$$V=\begin{pmatrix} a & 0 & 0 & d & 0 & 0 \\ 0 & b & c & 0 & -e & 0 \\ 0 & d & -a & 0 & 0 & 0 \\ c & 0 & 0 & -b & 0 & e \\ 0 & -e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \end{pmatrix}$$

$$a=-\mathrm{i}/3m^2, \qquad b=\mathrm{i}(D^1+\mathrm{i}D^2)(D^1-\mathrm{i}D^2)/3m^4$$

$$c=(D^1+\mathrm{i}D^2)/3m^3, \qquad d=-(D^1-\mathrm{i}D^2)/3m^3, \qquad e=\mathrm{i}/m.$$

Then, we get the required matrix by carrying out the coordinate transformation into the original system. The brackets with double asterisks defined by (13) are nothing but the final Dirac PB's. Thus the equal-time anticommutation relations among $\psi_{\mu\alpha}(x)$ and $\psi^\dagger_{\nu\beta}(y)$ are given by

$$\begin{aligned}\{\psi_{i\alpha}(x),\psi^\dagger_{j\beta}(y)\} &= \mathrm{i}[\mathrm{i}\gamma_j\gamma_i/2-(\mathrm{i}e/3m^3)(D_i+\mathrm{i}m\gamma_i/2)\tilde{R}\gamma^l\gamma_j\gamma^m F_{lm} \\ &\quad +(\mathrm{i}e/3m^3)\gamma^l\gamma_i\gamma^m F_{lm}\tilde{R}(D_j-\mathrm{i}m\gamma_j/2) \\ &\quad +(\mathrm{i}/3m^2)(D_i+\mathrm{i}m\gamma_i/2)R(D_j-\mathrm{i}m\gamma_j/2) \\ &\quad +(\mathrm{i}/3m^3)(2e/3m^2)(D_i+\mathrm{i}m\gamma_i/2)D^l\gamma^m F_{lm}(D_j-\mathrm{i}m\gamma_j/2) \\ &\quad +(\mathrm{i}/3m^4)(D_i+\mathrm{i}m\gamma_i/2)\tilde{R}\{(2e/3m^2)^2D^lD^mF_{lk}F^k_m-3m^2/2\} \\ &\quad \times(D_j-\mathrm{i}m\gamma_j/2)]_{\alpha\beta}\delta^3(x-y)\end{aligned} \tag{23}$$

$$\{\psi_{0\alpha}(x),\psi^\dagger_{j\beta}(y)\}=-(\tfrac{1}{2}\gamma_0 R\Gamma^i+\gamma_0\tilde{R}\Lambda^i)_{\alpha\gamma}\{\psi_{i\gamma}(x),\psi^\dagger_{j\beta}(y)\}$$

$$\{\psi_{0\alpha}(x),\psi^\dagger_{0\beta}(y)\}=-(\tfrac{1}{2}\gamma_0 R\Gamma^i+\gamma_0\tilde{R}\Lambda^i)_{\alpha\gamma}\{\psi_{i\gamma}(x),\psi^\dagger_{j\delta}(y)\}(\tfrac{1}{2}\Gamma^j R\gamma_0+\Lambda^{j\dagger}\tilde{R}\gamma_0)_{\delta\beta}$$

and the others are zero. It is also shown that the Heisenberg equations for $\psi_{i\alpha}(x)$ coincide with the Lagrangian equations and the adequacy of anticommutation relations is confirmed.

The characteristic differences between the cases of $\det R\neq 0$ and $\det R=0$ are in (i) the equal-time anticommutation relation $\{\psi_{i\alpha}(x),\psi^\dagger_{j\beta}(y)\}$ contains the second-order derivatives of $\delta^3(x-y)$ in the former case, while it contains the fourth-order ones in the latter case, (ii) the number of degrees of freedom of $\psi_{\mu\alpha}(x)$ is eight in the former case, while it reduces to six owing to the singular nature of R.

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